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Edge-disjoint odd cycles in planar graphs

Daniel Král'[a,1] and Heinz-Jürgen Voss[b]

[a] *Department of Applied Mathematics and Institute for Theoretical Computer Science, Charles University, Malostranské náměstí 25, 118 00 Prague 1, Czech Republic*

[b] *Institute of Algebra, Technical University, Dresden, Germany*

Abstract

We prove $\tau_{\text{odd}}(G) \leqslant 2\nu_{\text{odd}}(G)$ for each planar graph G where $\nu_{\text{odd}}(G)$ is the maximum number of edge-disjoint odd cycles and $\tau_{\text{odd}}(G)$ is the minimum number of edges whose removal makes G bipartite, i.e. which meet all the odd cycles. For each k, there is a 3-connected planar graph G_k with $\tau_{\text{odd}}(G_k) = 2k$ and $\nu_{\text{odd}}(G_k) = k$.

Keywords: Bipartite graphs; Odd cycles; Planar graphs; Erdős-Pósa property

1. Introduction

The goal of a feedback vertex (edge) problem in a graph is to find a set of vertices (edges) such that each cycle contains at least one of them. This problem is well studied in the general case where one wants to meet all the cycles both from the structural and algorithmic point of view, see [5] for a survey. We study relations among the size $\tau_{\text{odd}}(G)$ of a minimum set of edges which meets all the odd cycles of a graph G and the size $\nu_{\text{odd}}(G)$ of a maximum set of edge-disjoint odd cycles in a graph G. Brass [19] conjectured that $\tau_{\text{odd}}(G) \leqslant 2\nu_{\text{odd}}(G)$ for all graphs G, but this conjecture turned out to be false [1]. In 1999, Reed[2] proved [9] that for each positive integer s, there exists a projective-planar graph G with $\tau_{\text{odd}}(G) = s$ and $\nu_{\text{odd}}(G) = 1$, i.e., the graph which does not contain two edge-disjoint odd cycles, but it is necessary to delete at least s of its edges to make it bipartite (Reed considered in [9] the vertex

E-mail addresses: kral@kam.mff.cuni.cz (D. Král'), voss@math.tu-dresden.de (H.-J. Voss).

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[2] Reed wrote in [9] that this particular property of the so-called Escher walls was pointed out to him by Lovász and Schrijver.

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version of this problem, but his graphs are cubic and thus his result translates also to the edge case). In other words, odd cycles in general graphs do not satisfy the Erdős-Pósa property,[3] i.e., $\tau_{\mathrm{odd}}(G)$ is not bounded by a function of $v_{\mathrm{odd}}(G)$ for general graphs. We remark that it was proved in [8] that each $2000(k+1)$-connected graph contains either $k+1$ vertex-disjoint odd cycles or $2k$ vertices meeting all the odd cycles (another result of this type was proved in [14]). On the other hand, further results of [9] directly imply that the number of vertices meeting all the odd cycles is a function of the maximum number of vertex-disjoint odd cycles for planar graphs. In addition, Berge and Reed [2] in 2000 proved that odd cycles in planar graphs also in the edge version of the problem satisfy the Erdős-Pósa property, i.e., $\tau_{\mathrm{odd}}(G)$ is bounded by a function of $v_{\mathrm{odd}}(G)$ for planar graphs G. However, their function grows very fast. In this paper, we prove that the Brass' conjecture is true for planar graphs:

Theorem 1. *Let G be a plane graph. Then* $\tau_{\mathrm{odd}}(G) \leqslant 2v_{\mathrm{odd}}(G)$.

The inequality is tight for infinitely many planar graphs G as stated in Theorem 2 of Section 4. The precise bound for the corresponding vertex version remains to be an open problem.

We briefly survey related results in the case that one wants to meet all the triangles of a graph (note that triangles in graphs satisfy the Erdős-Pósa property by trivial arguments): Let $\tau_t(G)$ be the size of a minimum set of edges which meets all the triangles and let $v_t(G)$ be the size of a maximum set of edge-disjoint triangles of G. Tuza in [15] proved that $\tau_t(G) \leqslant 2v_t(G)$ for planar graphs and the factor 2 is the best possible one for this class of graphs. He conjectures that the inequality $\tau_t(G) \leqslant 2v_t(G)$ should hold for all graphs G. Tuza's proof was extended to a class of graphs which do not contain a subdivision of $K_{3,3}$ by Krivelevich in [7]. However, the original conjecture of Tuza remains still open.

Since we want to minimize the number of edges which meet all the odd cycles, we look for a maximum (in terms of the number of edges) bipartite subgraph of a graph, i.e., a maximum cut. For planar graphs, there is a nice connection between such sets of edges and solutions of a certain T-join problems explored by Hadlock in [6]; we describe this connection in Section 2. In particular, it is possible to determine the number $\tau_{\mathrm{odd}}(G)$ for planar graphs in polynomial time [6]. This problem is NP-hard for general graphs, see [20].

We use the standard graph notation throughout the paper and we refer the reader to any graph theory textbook if necessary. $N(v)$ denotes the set of neighbors of a vertex v including v itself. $G[U]$ is a subgraph of G induced by the vertices of U, $U \subseteq V(G)$. The graph G/E is a minor of G obtained by contracting the edges of E. We write $\tau_{\mathrm{odd}}(G)$ for the size of a minimum set of edges which meets all the odd

[3] The name of this property comes from the paper [4] by Erdős and Pósa. They proved by probabilistic arguments for general graphs that the minimum number of vertices whose deletion makes a graph acyclic is bounded by a function of the number of vertex-disjoint cycles. Constructive proofs of this are given in [13,16]. Other related results can be found in [17,18].

cycles of a graph G and $\nu_{\text{odd}}(G)$ for the size of the maximum set of edge-disjoint odd cycles in a graph G. We mean by a planar graph a graph which can be embedded in the plane and by a plane graph a fixed embedding of a planar graph. We say that a face of a plane graph is odd if the number of edges on its boundary is odd (counting bridges twice), in particular, it contains an odd cycle (cf. Lemma 1).

The paper is structured as follows: We recall basic concepts of linear programming and its relation to the feedback edge set problem for odd cycles in planar graphs in Section 2. We present the bound on $\tau_{\text{odd}}(G)$ in terms of $\nu_{\text{odd}}(G)$ for planar graphs G in Section 3 and we prove that it is attained by infinitely many planar graphs in Section 4. We conclude in Section 5.

2. Relation of linear programming to odd cycles in planar graphs

2.1. T-join problem and maximum cuts in planar graphs

Hadlock explored in [6] a nice connection between a certain T-join problem and finding a minimum set of edges in a planar graph whose removal makes the planar graph bipartite. We present this connection in this subsection.

In a T-join problem, you are given a connected graph H and an even-cardinality subset T of vertices of H. The goal is to find a subgraph H' of H with the least number of edges such that the set T consists of precisely the vertices of odd degrees in H'. Any subgraph H' of H whose odd-degree vertices are precisely the vertices of T is said to be a *solution* of the T-join problem and such a subgraph with the least number of edges is said to be an *optimal solution*. Fix a plane graph G and let G^* be its dual (multi)graph. Set T to be the set of all odd-degree vertices of G^*. Let E^* be any set of the edges of G^* and E edges of G corresponding to the edges of E^*. The graph $G \setminus E$ is bipartite if and only if G^*/E^* (as the dual of $G \setminus E$) has only vertices of even degrees. Hence, optimal solutions E^* of the just introduced T-join problem one-to-one correspond to cardinality-minimum sets of the edges E of G such that $G \setminus E$ is bipartite.

The just introduced connection between the two problems led to a polynomial-time algorithm for a maximum cut problem for planar graphs [6]. A T-join problem can be reduced to a weighted perfect matching problem. In the following, we widely use well-known statements from combinatorial optimization (in particular, from linear programming); we refer the reader to [3,11] for comprehensive introduction to this area.

Let H be a fixed graph and T a fixed subset of its vertices in the rest of this subsection. Form a complete graph K on the vertex set T and set the weight of the edge joining the vertices u and v ($u, v \in T$) to the length $d(u, v)$ of a shortest path between u and v in H. The weight of a minimum weight perfect matching is precisely the number of edges of an optimal solution of the original T-join problem. An optimal solution of the T-join problem may be formed by shortest paths corresponding to the edges included in a minimum weight perfect matching; such shortest paths are always edge-disjoint.

Let $O(T)$ be the set of all the odd-cardinality subsets of $T = V(K)$ of size three and more, i.e., $O(T) = \{o | o \subseteq T \ \& \ |o| \text{ is odd}\}$. We mean for the sake of brevity by an *odd subset* a subset with an odd number of vertices in the rest of the paper. The weight of the minimum weight perfect matching can be computed using the following linear program (the variables of the program correspond to the edges of K, i.e., the pair of u and v in x_{uv} is meant to be unordered):

$$\begin{aligned}
\sum_{e:e=uv,u\in T} x_e &= 1 \quad \text{for each } v\in T,\\
\sum_{e:e=uv,u\in o\&v\in T\setminus o} x_e &\geqslant 1 \quad \text{for each } o\in O(T),\\
x_e &\geqslant 0 \quad \text{for each } e\in E(K),
\end{aligned} \tag{1}$$

$$\min \sum_{u,v\in T} x_{uv}\, d(u,v).$$

There always exists an integral solution of this linear program, i.e., a solution with $x_e \in \{0,1\}$ for all $e\in E(K)$, which achieves the optimal value; the edges e with $x_e = 1$ then form a perfect matching of K of minimum weight and they correspond to shortest paths of an optimal T-join solution.

The dual linear program to (1) is the following:

$$\begin{aligned}
y_u + y_v + \sum_{o:|\{u,v\}\cap o|=1\&o\in O(T)} y_o &\leqslant d(u,v) \quad \text{for each } uv\in E(K),\\
y_o &\geqslant 0 \quad \text{for each } o\in O(T),
\end{aligned} \tag{2}$$

$$\max \sum_{v\in T} y_v + \sum_{o\in O(T)} y_o.$$

The optimal value $\max \sum_{v\in T} y_v + \sum_{o\in O(T)} y_o$ is equal to the weight of the minimum perfect matching of K (i.e., to the optimal value of the primal linear program) due to the duality of linear programming. We recall some properties which an optimal solution of this dual program may be assumed to have [12,3]. The dual solution y is *nested* if for any $o_1, o_2 \in O(K)$ the inequalities $y_{o_1} > 0$ and $y_{o_2} > 0$ imply that it holds either $o_1 \cap o_2 = \emptyset$ or $o_1 \subset o_2$ or $o_2 \subset o_1$. The dual solution y is *half-integral* if $2y_v$ and $2y_o$ are integers for all $v\in T$ and $o\in O(T)$. Let x be a fixed integral solution of the primal program. Then there exists an optimal non-negative nested half-integral dual solution y which moreover satisfies the following property (let us call this property the *blossom property* in the rest):

- Let $o_0 \in O(V(K)) = O(T)$ be an inclusion-wise minimal non-zero dual variable corresponding to set of $O(V(K))$, i.e., $y_o = 0$ for all $o \subset o_0, o\in O(V(K))$. Then: Exactly $(|o_0| - 1)/2$ of variables x_{uv} for $u, v\in o_0$ are equal to 1 (the remaining variables are equal to 0) and exactly one of variables x_{uv} for $u\in o_0, v\in V(K)\setminus o_0$ is equal to 1 (the remaining variables are again equal to 0). Moreover, the graph $K[o_0]$ contains a cycle of length $|o_0|$ such that $w(uv) = y_u + y_v$ for any two neighbors u and v in this cycle.

If we contract the vertices of o_0 in K to a single vertex a and set x', y' and w' as follows:

$$\begin{array}{ll}
w'(uv) = w(uv) & \text{for } u, v \notin o_0, \\
w'(ua) = \min\limits_{v \in o_0} w(uv) - y_v & \text{for } u \notin o_0, \\
x_{uv}' = x_{uv} & \text{for } u, v \in V(K) \backslash o_0, \\
x_{ua}' = 0 & \text{for } u \in V(K) \backslash o_0, \\
 & \text{if } x_{uv} = 0 \text{ for all } v \in o_0, \\
x_{ua}' = 1 & \text{for } u \in V(K) \backslash o_0, \\
 & \text{if } x_{uv} = 1 \text{ for some } v \in o_0, \\
y_v' = y_v & \text{for } v \in V(K) \backslash o_0, \\
y_a' = y_o & \\
y_o' = y_o' & \text{for } o \in O(V(K) \backslash o_0 \cup a) \text{ with } a \notin o, \\
y_o' = y'_{(o \backslash a) \cup o_0} & \text{for } o \in O(V(K) \backslash o_0 \cup a) \text{ with } a \in o.
\end{array}$$

Then the pair x' and y' is a pair of optimal primal and dual solutions corresponding to a minimum weighted perfect matching problem in K' with the weight function w' which is non-negative, nested and half-integral and it has the blossom property.

A pair of the optimal primal solution x and the optimal dual solution y in case of a T-join problem in a plane graph has a nice geometric interpretation which we describe in the next subsection.

2.2. Concept of "control zones"

We introduce a concept of "control zones" in the dual graph; this concept visualizes optimal non-negative half-integral nested dual solutions y of the T-join problem. This concept is inspired by the concept of "moats" known from the perfect matching problem for points in the plane.

Fix a plane graph G (let G^* be again its dual graph and T the set of all odd-degree vertices of G^*) together with a pair of primal and dual optimal solutions x and y such that y is non-negative, nested and half-integral and it has the blossom property. We define a *distance of an edge e to a vertex v* to be the length of the shortest path starting at v whose last edge is e, i.e., the distance of an edge e incident to a vertex v from v is 1. The control zones consisting of the edges of G^* will be created for vertices of T and odd-cardinality subsets o of T such that $y_o > 0$. We assign to each vertex $v \in T$ all edges of G^* at distance at most $\lfloor y_v \rfloor$ from v in G^*; if y_v is not integral, we assign to the vertex v also the (closer) "halves" of the edges at distance $\lceil y_v \rceil$ from v (Fig. 1). Thus, we have created a control zone for each vertex of T. Note that in case that $y_v = 0$, the control zone of v is empty. The control zone of $o \in O(T)$ contains an edge e of G^* if e is not contained in the union of the control zones for $v \in o$ and for $o' \subset o, o' \in O(T)$ and if the distance of e from some vertex v of o is at most $\lfloor y_v + \sum_{o': v \in o', o' \subset o, o' \in O(T)} + y_o \rfloor$; if such an edge e is contained in the union of the

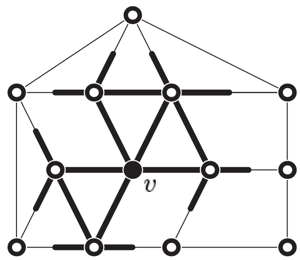

Fig. 1. A control zone (bold edges) of a vertex v (drawn by a full circle) with $y_v = 3/2$.

control zones only partially (i.e., only a half of it is contained), the remaining half of e is contained in the control zone of o. The control zone of $o \in O(T)$ contains also the closer halves of the edges e at distance at most $\lceil y_v + \sum_{o':v \in o', o' \subset o, o' \in O(T)} + y_o \rceil$ from some vertex $v \in o$ if the particular half of e is not contained in the union of the control zones for $v \in o$ and for $o' \subset o, o' \in O(T)$. Hence the union of control zones for $v \in o$, for $o' \subset o$ $(o' \in O(T))$ and for o contains fully all the edges at distance at most $\lfloor y_v + \sum_{o':v \in o', o' \subset o, o' \in O(T)} + y_o \rfloor$ from some $v \in o$ as well as closer halves of the edges at distance $\lceil y_v + \sum_{o':v \in o', o' \subset o, o' \in O(T)} + y_o \rceil$ from some $v \in O$ if this expression is not an integer.

The important property is that each edge of G^* is included to at most one control zone or is halved to two different control zones: This is true for a pair of control zones corresponding to $v \in T$ and $o \in O(T)$ such that $v \in o$ and for a pair corresponding to o and o' $(o, o' \in O(T))$ such that $o' \subset o$ due to the definition. For the remaining pairs of v and o and pairs o and o', this is true due to the following inequality from (2) (the first inequality rewritten to a more suitable form):

$$y_u + \sum_{o:u \in o, v \notin o, o \in O(T)} y_o + y_v + \sum_{o:v \in o, u \notin o, o \in O(T)} y_o \leqslant d(u, v)$$
$$\text{for each } u, v \in T.$$

We sketch how this fact can be derived from the above inequality: Assume that the control zones of o_1 and o_2 have a non-empty intersection; let us say they have a whole edge e in common. Then there exist vertices $v_1 \in o_1$ and $v_2 \in o_2$ such that the distance of e from v_1 is at most $\lfloor y_{v_1} + \sum_{o:v_1 \in o, o \subseteq o_1, o \in O(T)} y_o \rfloor$ from v_1 and $\lfloor y_{v_2} + \sum_{o:v_2 \in o, o \subseteq o_2, o \in O(T)} y_o \rfloor$ from v_2. But then the sum of these two expressions is at least $d(v_1, v_2) + 1$. Since $o_1 \cap o_2 = \emptyset$ (the dual solution is nested) and the dual solution is non-negative, we have:

$$y_{v_1} + \sum_{o:v_1 \in o, v_2 \notin o, o \in O(T)} y_o \geqslant y_{v_1} + \sum_{o:v_1 \in o, o \subseteq o_1, o \in O(T)} y_o,$$

$$y_{v_2} + \sum_{o:v_2 \in o, v_1 \notin o, o \in O(T)} y_o \geqslant y_{v_2} + \sum_{o:v_2 \in o, o \subseteq o_2, o \in O(T)} y_o.$$

This immediately contradicts that y is a solution of (2):

$$y_{v_1} + \sum_{o:v_1 \in o, v_2 \notin o, o \in O(T)} y_o + y_{v_2} + \sum_{o:v_2 \in o, v_1 \notin o, o \in O(T)} y_o \geqslant d(v_1, v_2) + 1.$$

Hence any two different control zones have an empty intersection. Through each control zone passes exactly one path of the solution of the T-join problem corresponding to x (due to the first part of the blossom property). The path between the pair of the vertices u and v of T in the solution of the T-join problem corresponding to x passes only through the control zones of u, v and o with $|o \cap \{u, v\}| = 1$ (this is also assured by the first part of the blossom property).

Actually, the just introduced control zones one-to-one correspond to non-negative half-integral and nested solutions of the dual program (2). More precisely, if y is non-negative, half-integral and nested then y is a feasible (not necessarily optimal) solution of the dual program (2) corresponding to a T-join problem in G^* if and only if the control zones corresponding to y are pairwise disjoint (note that in order to construct the control zones, it was only necessary that y is non-negative, half-integral and nested). Moreover, if there exists a set of $|T|/2$ edge-disjoint paths joining pairs of vertices of T such that the path between u and v passes only through the control zones of u, v and o with $|o \cap \{u, v\}| = 1$ and through each of the control zones at most one of the path passes, then the dual solution y corresponding to the control zones is optimal (it is possible to find an integral solution x corresponding to this set of paths and the duality of linear programming assures the optimality of y). We implicitly use this correspondence (same as in the case of moats from [3, Chapter 5]) during the proof of Theorem 1 without emphasizing this fact.

3. Proof of Theorem 1

We first state an easy lemma which we use during the proof of Theorem 1.

Lemma 1. *A boundary of any odd face of a plane graph contains an odd cycle.*

Proof. Let E be the set of edges on the boundary of the face without the bridges. Since the face is odd and the bridges are counted twice, $|E|$ is odd. The edges of E form an even-degree subgraph H of the plane graph. H can be partitioned into 2-factors, i.e., into cycles. At least one of these cycles contains an odd number of edges and it is actually an odd cycle. $\square$

We give an intuitive explanation of the proof of Theorem 1 before an accurate proof. Let G be a plane graph, G^* its dual graph and T the set of odd-degree vertices of G^*. Let x and y be the optimal primal and dual solution corresponding to the T-join problem where y is non-negative, half-integral and nested, and it has the blossom property. We find $\lfloor y_v \rfloor$, $v \in T$, edge-disjoint odd cycles formed by the edges of G corresponding to the edges of G^* joining a vertex at distance i to a vertex at distance $i+1$ from v in G^* for $0 \leqslant i \leqslant \lfloor y_v \rfloor - 1$ ("concentrical" cycles around the face corresponding to v). We take additional care when y_v is not integral and two different control zones share the same edge(s). In a fashion similar to finding edge-disjoint odd cycles around vertices v, $v \in T$, we find edge-disjoint odd cycles around sets o with $y_o > 0$, $o \in O$. In the formal proof of the theorem, we proceed by induction on the number of faces and we find edge-disjoint odd cycles one by one.

Proof of Theorem 1. Let G be a plane graph, G^* its dual graph and T the set of odd-degree vertices of G^*. Fix x and y optimal primal and dual solutions of the T-join problem such that y is non-negative, half-integral and nested, and it has the blossom property. The proof proceeds by induction on the number of faces of G. Assume that G is bridgeless but we allow G to be disconnected. Note that G^* may actually be a multigraph but it is loopless and bridgeless. Let $\mathscr{S}(y)$ be the following expression:

$$\mathscr{S}(y) := \frac{\sum_{u \in T} y_u + \sum_{u \in T} \lfloor y_u \rfloor + \sum_{o \in O} y_o}{2}.$$

We construct a set M of at least $\mathscr{S}(y)$ edge-disjoint odd cycles of G. Since $\tau_{\text{odd}}(G) = \sum_{u \in T} y_u + \sum_{o \in O} y_o \leqslant 2\mathscr{S}(y)$, the number of the constructed edge-disjoint odd cycles of G is certainly at least $\tau_{\text{odd}}(G)/2$. Moreover, the cycles in M will satisfy the following condition (C_u is the boundary of the face corresponding to a vertex u in G^*):

($*$) For each $u \in T$ with $y_u \geqslant 1$, there is exactly one odd cycle in M completely contained in C_u and this cycle is the only one from M which is not edge-disjoint with C_u.

We distinguish three cases which cover all the possibilities:

- *There exists a vertex $v \in T$ with $y_v > 1$.*

  Let E be the union of all C_u, $u \in N(v)$, and let E' be their symmetric difference (i.e., E' contains those edges contained in an odd number (which has to be exactly one in this case) of C_u for $u \in N(v)$). Let G' be the graph obtained from G by removing the edges of $E \backslash E'$. This corresponds to a contraction of the set $N(v)$ of G^* to the vertex v (with simultaneous removal of arising loops)—see Fig. 2. For convenience, let a denote the vertex after contracting $N(v)$. The boundary of the newly created face is E'. Note that $C_v \subseteq E \backslash E'$. The vertex v is the only vertex of

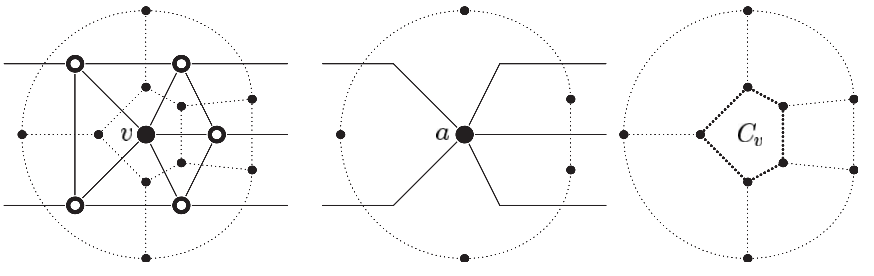

Fig. 2. The reduction in the case $y_v > 1$. The vertex v is drawn by a full circle. The edges of G^* are solid and the edges of G are dotted. The odd cycle of C_v is marked in the left figure.

$N(v)$ of odd-degree (otherwise the first inequality of (2) would be violated for v and its odd-degree neighbor because y is non-negative). Hence $|E'|$ is odd and the newly created face is an odd face. Moreover, all the (edge-disjoint) paths of the T-join except for the path from v are vertex-disjoint with $N(v)$ (at least half of any edge incident with a vertex of $N(v)$ is included in the "control zone" of v since $y_v \geqslant 3/2$). We set $y'_a := y_v - 1$, $y'_u := y_u$ for $u \neq v$, $u \in T$, and $y'_o := y_o$, $o \in O$ (identifying the vertex a with v in the sets $o \in O$). We obtain a feasible solution of the dual problem for G'. This decreases $\mathscr{S}(y)$ by 1: $\mathscr{S}(y') = \mathscr{S}(y) - 1$. The primal solution of the original problem is also a primal solution of the new problem (if we identify the vertex v of G with the vertex a of G') and its weight is decreased by 1. Since the value of the primal solution is equal to the value of the solution of the dual problem, it is an optimal one (thus $\tau_{\mathrm{odd}}(G) = \tau_{\mathrm{odd}}(G') + 1$). Hence we can use induction. We can always extend a set M' of at least $\mathscr{S}(y') = \mathscr{S}(y) - 1$ odd cycles of G' to a set M of at least $\mathscr{S}(y)$ odd cycles of G by adding an odd cycle contained in C_v (it exists due to Lemma 1). If the cycles of M' in G' satisfy condition $(*)$, then the cycles of M in G satisfy condition $(*)$, too.

- *It holds that $y_v \leqslant 1$ for each $v \in T$ and there exists a set $o \in O(T)$ with $y_o > 0$.*

  Choose a minimal (by inclusion) o with $y_o > 0$. As stated in Section 2, the set o contains in the auxiliary complete graph K on the vertices T a cycle of length $|o|$ consisting of edges $e = uv \in E(K)$ such that $w(e) = y_u + y_v$. Let $v_1 v_2 \ldots v_n$ be this cycle in K (i.e. $n = |o|$ and $o = \{v_1, \ldots, v_n\}$). We have $d(v_i, v_{i+1}) = y_{v_i} + y_{v_{i+1}}$, $1 \leqslant i \leqslant n$ (indices are taken modulo n and the distance is measured in G^*) due to the blossom property. Since $y_v \leqslant 1$ for each $v \in T$, either all y_{v_i} are equal to $1/2$ or all $y_{v_i} \in \{0, 1\}$ for $1 \leqslant i \leqslant n$. Hence $d(v_i, v_{i+1}) \in \{1, 2\}$ for $1 \leqslant i \leqslant n$ (indices are again taken modulo n). There is exactly one vertex a of o which is matched to a vertex outside o in K due to the blossom property.

  In the case that all y_{v_i} for $1 \leqslant i \leqslant n$ are integral, we proceed as follows: Let $W = \bigcup N(v)$ where the union is taken over all vertices $v \in o$ with $y_v = 1$. If $y_v = 0$ for $v \in o$, then both the neighbors v' and v'' of v in the cycle in K have $y_{v'} = 1$ and $y_{v''} = 1$. Then v' and v'' are neighbors of v in G^* and $v \in W$. Hence $o \subseteq W$ and the subgraph $G^*[W]$ is connected. Let E be the union of C_v, taken over all $v \in W$, and let E' be the symmetric difference of C_v (E' contains those edges which are contained in an odd number of C_v), taken over all $v \in W$. Let G' be the graph obtained from G by removing the edges of $E \backslash E'$. Since $G^*[W]$ is connected, this corresponds to a contraction of the set W in G^* to the vertex a (with simultaneous removal of arising loops). The boundary of the newly created face is E'. Note that $C_{v_i} \subseteq E \backslash E'$ for $y_{v_i} = 1$, $1 \leqslant i \leqslant n$ (we do not say anything about C_{v_i} with $y_{v_i} = 0$). Since the only odd-degree vertices of G contained in W are $v_1, \ldots, v_n$ (the distance between any $u \in T \backslash o$ and $v \in o$ with $y_v = 1$ has to be at least $y_v + y_o \geqslant 3/2$ due to (2)), the newly created face of G is odd and the degree of a in G^* is odd. Moreover, the only (edge-disjoint) paths between vertices of T, which are not vertex-disjoint with W, are the paths from or to the vertices of o. Hence the contraction of the set W in G^* is exactly an application of the blossom property for the set o in the auxiliary complete graph. This gives us the pair of optimal primal and dual solutions x' and y' which form optimal solutions for the problem in G'. This decreases $S(y)$ by at most $\sum_{1\leqslant i\leqslant n} y_{v_i}$, i.e., $\mathscr{S}(y')\geqslant\mathscr{S}(y)-\sum_{1\leqslant i\leqslant n} y_{v_i}$. We use induction. A set M' of at least $\mathscr{S}(y')$ odd edge-disjoint cycles of G' can always be extended by adding the odd edge-disjoint cycles contained in C_{v_i} for $y_{v_i}=1$, $1\leqslant i\leqslant n$, (their existence follows from Lemma 1). This gives a set M of at least $\mathscr{S}(y')+\sum_{1\leqslant i\leqslant n} y_{v_i}\geqslant\mathscr{S}(y)$ odd edge-disjoint cycles of G. Note that the conditions $y_{v_i}=y_{v_j}=1$ for $i\neq j$ imply that each pair of C_{v_i} and C_{v_j} is edge-disjoint and each C_{v_i} with $y_{v_i}=1$ is fully contained in $E\backslash E'$. If the cycles of M' in G' satisfy the condition $(*)$, then the cycles of M in G also satisfy the condition $(*)$.

Next, we deal with the remaining case that $y_{v_i}=1/2$ for all $1\leqslant i\leqslant n$ in this paragraph. Let $W:=o$ be the set consisting of all v_i for $1\leqslant i\leqslant n$. Let E be the union of C_v, taken over all $v\in W$, and E' the symmetric difference of C_v (E' contains those edges which are in an odd number of C_v), taken over all $v\in W$. The graph $G^*[W]$ is obviously connected because $v_1v_2\ldots v_n$ is a cycle of G^*. Let G' be the graph obtained from G by removing the edges of $E\backslash E'$. This corresponds to a contraction of the set W to the vertex a (with simultaneous removal of arising loops)—see Fig. 3. The boundary of the newly created face is E'. Since W consists of n odd-degree vertices of G^*, the newly created face of G is odd and the degree of a in G^* is odd. This contraction again corresponds to an application of the blossom property to the set o in the auxiliary graph except that we have to set ${y_a}'=y_o+y_a=y_o+1/2$ (instead of ${y_a}'=y_o$) and the weights of the edges incident with a in the auxiliary complete graph are larger by $1/2$ (compared to the weights obtained by the application of the blossom property to the set o). The primal solution x' and the dual solution y' (with the above described modification) are optimal solutions of the problem in G'. The value of $S(y)$ is decreased by $n/4+y_o/2$ due to the removal of y_{v_i}, for $1\leqslant i\leqslant n$, and y_o from the sum and it is increased by $1/4+y_o/2$ due to adding ${y_a}'$ to the sum and additional at least $1/2$ due to the fact that ${y_a}'=y_o+1/2\geqslant 1$. We conclude that $S(y')\geqslant S(y)-(n-3)/4$. Moreover, we have $\tau_{\text{odd}}(G)=\tau_{\text{odd}}(G')-1/2+\sum_{1\leqslant i\leqslant n} y_{v_i}=\tau_{\text{odd}}(G')+(n-1)/2$. Note that $C_{v_i}\subseteq E$, but it might be that $C_{v_i}\cap(E\backslash E')\neq\emptyset$ for all $1\leqslant i\leqslant n$. The boundaries C_{v_i} and C_{v_j} are not disjoint iff v_i and v_j are adjacent in G. Since G^*

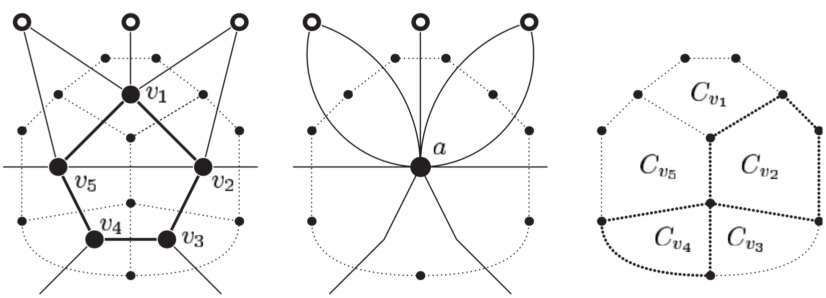

Fig. 3. The reduction in the case $y_{v_i}=1/2$ for all $v_i\in o=\{v_1,v_2,v_3,v_4,v_5\}$, $y_o>0$. The vertices v_i are drawn by full circles. The edges of G^* are solid and the edges of G are dotted. The odd cycles of the boundaries C_{v_2} and C_{v_4} are marked in the left figure.

is planar, there are at least $n/4$ mutually non-adjacent vertices among $v_1, \ldots, v_n$ due to the four-color theorem. Hence at least $(n+1)/4$ of the boundaries C_{v_i} (recall that n is odd) are edge-disjoint and each of them contains an odd cycle due to Lemma 1. It is the right time to use induction. G' contains at least $\mathscr{S}(y')$ edge-disjoint odd cycles such that exactly one of them is not disjoint with C_a in G' (recall that ${y_a}' \geqslant 1$). If we replace this odd cycle with at least $(n+1)/4$ edge-disjoint odd cycles mentioned above, we get at least $\mathscr{S}(y') + (n+1)/4 - 1$ edge-disjoint odd cycles of G. A simple calculation gives the following bound:

$$\mathscr{S}(y') + (n+1)/4 - 1 = \mathscr{S}(y') + (n-3)/4 \geqslant \mathscr{S}(y)$$

Hence, we get at least $\mathscr{S}(y)$ edge-disjoint odd cycles of G which satisfy condition $(*)$.

- *It holds that $y_v \leqslant 1$ for each $v \in T$ and $y_o = 0$ for each $o \in O(T)$.*

  Let W be the set of all the vertices v with $y_v > 0$. Recall that $G^*[W]$ is the subgraph of G^* induced by the vertices of W. Note that each vertex v, $y_v = 1$, is an isolated vertex of $G^*[W]$, since the distance between v and any other vertex u with $y_u > 0$ has to be at least $y_v + y_u > 1$. Let n_1 be the number of vertices v with $y_v = 1$ and n_2 the number of vertices v with $y_v = 1/2$. The graph $G^*[W]$ contains an independent set A of size $n_1 + \lceil n_2/4 \rceil$ (there are n_1 isolated vertices and the rest of $G^*[W]$ contains an independent set of size at least $\lceil n_2/4 \rceil$ due to the four-color theorem). The boundaries C_v and C_w for $v, w \in A$, $v \neq w$, are edge-disjoint. Each C_v contains an odd cycle due to Lemma 1. The value $\mathscr{S}(y)$ is equal to $(2n_1 + n_2/2)/2 = n_1 + n_2/4$ and hence we have enough cycles. Since we include among these cycles exactly one cycle fully contained in C_v for each v, $y_v \geqslant 1$, the cycles also satisfy condition $(*)$. $\square$

4. Tightness of the bound

Theorem 2. *There is a 3-connected planar graph G with $\tau_{\mathrm{odd}}(G) = 2k$ and $\nu_{\mathrm{odd}}(G) = k$ for each integer $k \geqslant 1$.*

Note that if we leave out the assumption that G is 3-connected, the theorem becomes trivial because one can consider a disjoint union of k copies of K_4.

Proof. Let k be the fixed integer from the statement of the theorem. Let G be a 3-connected quadrangulation with at least k vertices of degree 3. We replace k vertices of degree three with the gadget from Fig. 4. Let G' be the obtained graph. Note that G' is 3-connected. We claim that $\tau_{\mathrm{odd}}(G') = 2k$ and $\nu_{\mathrm{odd}}(G') = k$. In order to destroy all odd cycles fully contained in one gadget, two edges of it have to be deleted. There are k disjoint gadgets in G'. Hence at least $2k$ edges have to be deleted to make G' bipartite. We conclude that $\tau_{\mathrm{odd}}(G') \geqslant 2k$.

On the other hand: If in each of the k gadgets the edges a and b (cf. Fig. 4) are deleted, the obtained graph is bipartite. Hence $\tau_{\mathrm{odd}}(G') \leqslant 2k$ and $\tau_{\mathrm{odd}}(G') = 2k$. This

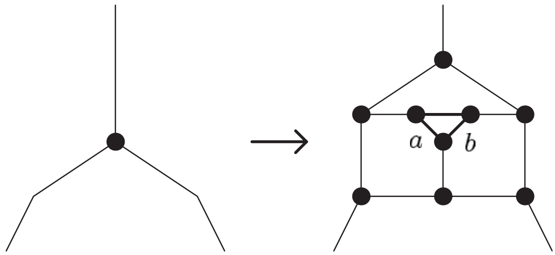

Fig. 4. Replacing vertices of degree three in a quadrangulation with the gadget.

also implies: Each odd cycle contains the edge a or the edge b from at least one of the gadgets. But then it includes at least two vertices from the inner triangle (the bold triangle in Fig. 4). Consequently, each odd cycle contains from at least one gadget two vertices of its inner triangle. Hence G' has no $k+1$ edge-disjoint odd cycles because otherwise by the pigeon-hole principle two of those odd cycles would contain two vertices of the inner triangle of the same gadget and would have a common vertex and, therefore, also a common edge (all the vertices of the inner triangle have degree 3). Thus $\nu_{\text{odd}}(G') \leqslant k$. Each of the k disjoint gadgets contains an odd cycle and hence $\nu_{\text{odd}}(G') \geqslant k$. The last inequality also follows from Theorem 1. $\square$

5. Conclusion

We proved that $\tau_{\text{odd}}(G) \leqslant 2\nu_{\text{odd}}(G)$ for any planar graph G. Our proof uses the four-color theorem to obtain an independent set of size at least $n/4$ in an n-vertex planar graph (which has neither been shown to be equivalent to the four color theorem nor proved independently of it). If we used instead the five-color theorem (whose proof is incomparable with a complex proof of the four-color theorem even after its recent refinement [10]), we would get the inequality $\tau_{\text{odd}}(G) \leqslant \frac{5}{2}\nu_{\text{odd}}(G)$. It might be interesting to find a proof of the sharp inequality $\tau_{\text{odd}}(G) \leqslant 2\nu_{\text{odd}}(G)$ which does not use the four-color theorem.

Reed's Escher walls from [9] provide an example of projective planar graphs G with $\tau_{\text{odd}}(G)$ arbitrary large and $\nu_{\text{odd}}(G) = 1$. But if $\tau_{\text{odd}}(G)$ is large, these graphs are not embeddable to a torus. Hence the following problem arises:

Problem 1. What is the relation between $\tau_{\text{odd}}(G)$ and $\nu_{\text{odd}}(G)$ for graphs G which can be embedded to a torus?

Or even more generally:

Problem 2. Describe (if there is any) the relation between the numbers $\tau_{\text{odd}}(G)$, $\nu_{\text{odd}}(G)$ and $g(G)$ for a graph G which can be embedded to an orientable surface of genus at most $g(G)$.

It could be probably derived from the results of [9] that $\tau_{\mathrm{odd}}(G)$ is bounded by a function of $v_{\mathrm{odd}}(G)$ and $g(G)$.

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